Minimum-Length MHD Accelerator with Constant Enthalpy and Magnetic Field

HAROLD MIRELS,* RICHARD R GOLD,† AND JAMES F MULLEN‡
Aerospace Corporation, El Segundo, Calif

IN a recent article, Drake[1] has presented a method for determining the minimum length of crossed-field MHD accelerators with specified entrance and exit conditions Drake has assumed that the temperature (which is proportional to enthalpy for a perfect gas) was constant throughout the accelerator and has developed a numerical procedure for finding the minimum length In many practical cases, the joule heating per unit volume is small compared with the net local electrical power input For these cases, it is possible to solve the minimum-length problem in closed form This solution is indicated herein for the case of constant enthalpy, magnetic field, and electrical conductivity A brief comparison is made with the constant enthalpy accelerator solution presented in Ref 2 The notation and configuration are the same as in Ref 2

The one-dimensional equations of motion, neglecting heat conduction, viscous effects, and ion slip, and assuming constant enthalpy, are as follows:

Energy

$$u(du/dx) = jE_yA \tag{1a}$$

Momentum

$$\rho u(du/dx) = -(dp/dx) + jB \tag{1b}$$

Continuity

$$\rho uA = 1 \tag{1c}$$

Ohm's Law

$$j = \sigma(E_y - uB) \tag{1d}$$

State (Perfect Gas)

$$p/p_1 = \rho \tag{1e}$$

Scalar Conductivity

$$\sigma = \sigma(\rho) \tag{1f}$$

Equations (1) are nondimensionalized mks equations, where $\rho = (\rho/\rho_1)^*$, $p = (p/\rho_1 u_1^2)^*$, $u = (u/u_1)^*$, $A = (A/A_1)^*$, $B = (B/B_1)^*$, $E_y = (E_y/u_1B_1)^*$, $j = (j/\sigma_1 u_1 B_1)^*$, $\sigma = (\sigma/\sigma_1)^*$, and $x = (\sigma_1 B_1^2 x/\rho_1 u_1^2)^*$ The superscript asterisk denotes dimensional quantities, and the subscript 1 denotes the value at the accelerator inlet ($x = 0$) Recall from Ref 2 that E_y, j are in the y direction, B is in the z direction, and u is in the x direction The accelerator is segmented so that there is no current in the x direction Note that $p_1 = 1/\gamma_1 M_1^2$ for an ideal gas

The ratio of the local joule heating to the local net electrical power input is [from Eq (1d)]

$$\frac{\text{joule heating}}{\text{net power input}} \equiv \frac{j^2/\sigma}{j^2/\sigma + jBu} = \frac{\beta - 1}{\beta} \tag{2}$$

where $\beta = E_y/uB$ It is desirable to have this energy ratio small in MHD accelerators, so that most of the electrical power goes directly into accelerating the fluid The value of $(\beta - 1)\beta$ at the accelerator inlet is denoted herein by ϵ Thus

$$\epsilon \equiv \left(\frac{\beta - 1}{\beta}\right)_1 \equiv \frac{E_{y,1} - 1}{E_{y1}} \tag{3}$$

Later, this parameter will be assumed to be small

Following Drake, we will now derive an integral expression for accelerator length which is to be minimized Equations (1a, 1c, and 1d) give

$$\frac{du}{dx} = \frac{\sigma B^2 \beta(\beta - 1)}{\rho} \tag{4}$$

Similarly, Eqs (1b, 1d, and 1e) give

$$\frac{du}{dx} = \frac{\sigma u B^2(\beta - 1)}{\rho u + p_1\rho'} \tag{5}$$

where $\rho' \equiv d\rho/du$ Equating Eqs (4) and (5) yields

$$p_1\rho'/\rho u = -(\beta - 1)/\beta \tag{6}$$

Substituting Eq (6) into Eq (4), for β, and integrating, then yields

$$x_2 = -\int_1^{u_2} \frac{1}{\sigma B^2}\frac{\rho u}{p_1\rho'}\left(1 + \frac{p_1\rho'}{\rho u}\right)^2 \rho du \tag{7}$$

Here, x_2 is the length of an accelerator having an exit velocity u_2 If B is considered a known function of ρ and u, then Eq (7) is of the form

$$x_2 = \int_1^{u_2} F(u,\rho,\rho')du$$

By applying the calculus of variations, it is possible to find the relation between ρ and u such that x_2 is a minimum This relation satisfies the Euler equation $\partial F/\partial\rho - d(\partial F/\partial\rho')/du = 0$ and can be found by integrating the latter

Consider the case $\sigma = B = 1$ (as was done by Drake) Substituting the integrand of Eq (7) into Euler's equation yields

$$\frac{dR}{du} = \frac{-R}{2u}(1 - \epsilon R)\left(1 + \epsilon R + \frac{2\epsilon}{p_1}Ru^2\right) \tag{8}$$

where

$$R \equiv -\frac{p_1\rho'}{\epsilon\rho u} \equiv \frac{(\beta - 1)/\beta}{[(\beta - 1)/\beta]_1} \tag{9}$$

It is seen that $R = 1$ when $u = 1$ so that ϵR is of order ϵ We now assume

$$\epsilon \ll 1 \tag{10}$$

Neglecting terms of order ϵR, compared with one, in Eq (8) yields

$$\frac{dR}{du} + \frac{1}{2u}R = -\frac{\epsilon}{p_1}uR^2 \tag{11}$$

The retention of the term on the right-hand side is justified only if ϵ/p_1 remains of order one as $\epsilon \to 0$ It will later be shown that $\epsilon/p_1 \geq 1$ for nonconverging channels Equation (11) is a Bernoulli differential equation and can be integrated, with initial conditions $R = u = 1$, to yield

$$1/R = u^{1/2}[1 + \tfrac{2}{3}(\epsilon/p_1)(u^{3/2} - 1)] \tag{12}$$

Note that R decreases with increasing u, so that the neglect of terms ϵR, compared with one, becomes a better approxi-

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* Head, Advanced Propulsion and Fluid Mechanics Department, Aerodynamics and Propulsion Laboratory Associate Fellow Member AIAA

† Head, Magnetogasdynamics Section, Aerophysics Department, Aerodynamics and Propulsion Laboratory Associate Follow Member AIAA

‡ Member, Technical Staff, Advanced Propulsion and Fluid Mechanics Department, Aerodynamics and Propulsion Laboratory Member AIAA

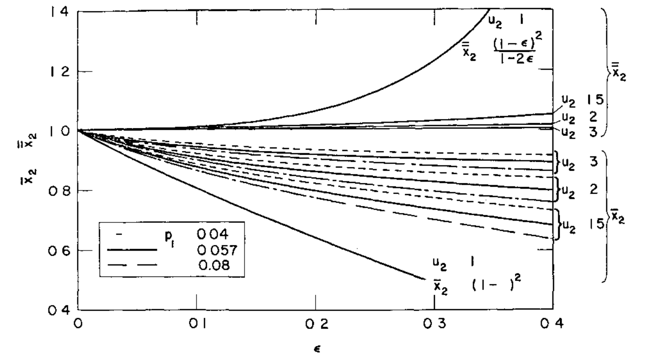

Fig 1 Ratio of accelerator length obtained from numerical integration [Eqs (7–9)] to length obtained from perturbation solution [Eqs (20) and (22)]

mation as we proceed down the channel § Substituting Eq (12) into the first equality of Eq (9), and integrating, gives

$$\rho = [1 + \tfrac{2}{3}(\epsilon/p_1)(u^{3/2} - 1)]^{-1} = u^{1/2}R \qquad (13)$$

which is the relation between ρ and u for a minimum accelerator length Omitting the subscript 2 in Eq (7), we find that the accelerator length x, as a function of u, is [noting Eqs (9) and (13)]

$$x = \frac{1}{\epsilon}\int_1^u (1 - \epsilon R)^2\, u^{1/2}du \qquad (14a)$$

$$= (2/3\epsilon)(u^{3/2} - 1) \qquad (14b)$$

The term ϵR was neglected, compared with one, in the integration of Eq (14a) All other variables can be found readily By definition, $E_y \equiv u/(1 - \epsilon R)$ Thus, retaining the leading terms, we find

$$E_y = u(1 + \epsilon R) \qquad (15a)$$

$$j = \epsilon R u \qquad (15b)$$

$$jE_yA = \epsilon u^{1/2} \qquad (15c)$$

Other quantities of interest, discussed in Ref 2, are

$$\omega_e\tau_e \equiv \left[\frac{\omega_e\tau_e}{(\omega_e\tau)_1}\right]^* = \frac{B}{\rho} = \frac{1}{\rho} \qquad (B = 1) \qquad (16a)$$

$$E_x \equiv \left[\frac{E_x}{u_1B_1(\omega_e\tau)_1}\right]^* = \frac{jB}{\rho} = \epsilon u^{1/2} \qquad (B = 1) \qquad (16b)$$

$$\Phi \equiv -\int_0^x E_x dx = p_1 \ln\left(\frac{1}{\rho}\right) - \frac{u^2 - 1}{2} \qquad (16c)$$

Equations (13) to (16) define the present solution

The variation of area with x is given by

$$A = \frac{1}{\rho u} = \frac{1 + (\epsilon/p_1)\epsilon x}{[1 + \tfrac{3}{2}\epsilon x]^{2/3}} \qquad (17a)$$

$$\frac{dA}{dx} = \frac{\epsilon A[(\epsilon/p_1 - 1) + \tfrac{1}{2}(\epsilon/p_1)\epsilon x]}{[1 + (\epsilon/p_1)\epsilon x][1 + \tfrac{3}{2}\epsilon x]} \qquad (17b)$$

It is seen from Eq (17b) that $\epsilon/p_1 \geq 1$ for $dA/dx \geq 1$ at $x = 0$ [This criterion also applies for ϵ large, as can be determined by differentiating Eq (1c)] Since most practical accelerator designs will require $dA/dx \geq 1$, the previous statement that generally $\epsilon/p_1 \geq 1$ is confirmed For $(\epsilon x)^2 \ll 1$ and $\epsilon x \gg 1$, Eq (17a) has the forms, respectively,

$$A \approx 1 + [(\epsilon/p_1) - 1]\epsilon x \qquad (\epsilon x)^2 \ll 1 \qquad (18a)$$

$$A \approx (\tfrac{2}{3})^{2/3}(\epsilon/p_1)(\epsilon x)^{1/3} \qquad \epsilon x \gg 1 \qquad (18b)$$

Hence, for large ϵx, the channel is concave

If the accelerator exit conditions u_2, ρ_2 are specified, the required value of ϵ/p_1 is [from Eq (13)]

$$\frac{\epsilon}{p_1} = \frac{3}{2}\frac{1/\rho_2 - 1}{u_2^{3/2} - 1} \qquad (19)$$

The minimum accelerator length, to achieve these conditions, is then [from Eq (14b)]

$$x_2 = \frac{4}{9p_1}\frac{(u_2^{3/2} - 1)^2}{1/\rho_2 - 1} \qquad (20)$$

As expected, the length x_2 decreases with a decrease in ρ_2 However, the Hall parameter at the exit, $(\omega\tau)_2 = 1/\rho_2$, should not exceed the value at which ion slip starts to occur This provides a lower limit on the value of ρ_2 and therefore a lower limit on the x_2 required to achieve a velocity u_2

Equations (7–9) have been numerically integrated, for several cases, and the results have been compared with the present perturbation solution The results for accelerator length x_2 are plotted in Fig 1 in terms of

$$\bar{x}_2 \equiv \frac{x_2}{(4/9p_1)[(u_2^{3/2} - 1)^2/(1/\rho_2 - 1)]} \qquad (21)$$

The numerator is the value of x_2 obtained from the numerical integration, for given ϵ, p_1, u_2, ρ_2, and the denominator is the corresponding value of x_2 indicated by the perturbation solution [Eq (20)] It is seen, from Fig 1, that the perturbation solution tends to overestimate x_2 as ϵ increases As previously noted, the error decreases with an increase in u^2/p_1 (for a given ϵ) Equation (14a) can be integrated in closed form with the term $(1 - \epsilon R)^2$ retained Neglecting terms of order $(\epsilon R)^2$, compared with one, we then find

$$\frac{x_2}{(4/9p_1)(u_2^{3/2} - 1)^2/(1/\rho_2 - 1)} = 1 - \frac{3p_1D}{u_2^{3/2} - 1} \times \left[\ln\frac{\zeta - 1}{(\zeta^2 + \zeta + 1)^{1/2}} + 3^{1/2}\tan^{-1}\frac{2\zeta + 1}{3^{1/2}}\right]_{\zeta = D}^{\zeta = Du_2^{1/2}} \qquad (22)$$

where $D = [1 - (u_2^{3/2} - 1)/(1/\rho_2 - 1)]^{-1/3}$ Although this procedure is not strictly justifiable [since the factor $u^{1/2}$, in the integrand of Eq (14a), neglects terms of order ϵ], it leads to considerably improved estimates for x_2 The exact value of x_2, divided by the value of x_2 indicated by Eq (22), is denoted by $\bar{\bar{x}}_2$ and is also plotted in Fig 1 It is seen that Eq (22) slightly underestimates x_2 but is correct to within 5% for $1\,5 \leq u_2 \leq 3$, $\epsilon \leq 0\,4$, and $p_1 \approx 0\,06$ Note that $\bar{x}_2 \to (1 - \epsilon)^2$ and $\bar{\bar{x}}_2 \to (1 - \epsilon)^2/(1 - 2\epsilon)$ as $u_2 \to 1$ Although the percent error in using Eq (20) or (22) to estimate x_2 is largest for u_2 near one, the magnitude of the error is small since $x_2 \to 0$ as $u_2 \to 1$

Reference 2 has presented an analytical solution for a constant-enthalpy accelerator In addition to constant enthalpy, it has been assumed therein that

$$jE_yA = \frac{\epsilon}{(1 - \epsilon)^2}u^n \qquad (23a)$$

$$A = \left[1 + \frac{1}{2}\frac{\epsilon/p_1 - 1}{(1 - \epsilon)^2}\epsilon x\right]^2 \qquad (23b)$$

The latter assumptions replace the present assumptions that $B = 1$ and that the relation between ρ and u should be that which yields a minimum length The solution of Ref 2 was used in a parametric study of an MHD wind tunnel and gave reasonable accelerator lengths, area variation, and electrical

§ The present approximation is valid for $R\epsilon \ll 1$ For large u, Eq (12) shows $\epsilon R \approx \epsilon/[\tfrac{2}{3}(\epsilon/p_1)u^2]$ Hence, the solution is valid, not only for small ϵ but also, asymptotically, for large $u^2/p_1 \equiv \gamma_1 M^2$ (regardless of ϵ)

energy input distributions The accelerator lengths were shorter than those obtained from several other physically realistic analytic solutions but obviously did not represent a mathematical minimum It is therefore of interest to compare the present solution with that in Ref 2

First, it can be seen that, for $\epsilon \ll 1$ and $(\epsilon x)^2 \ll 1$, both solutions are identical provided $n = \frac{1}{2}$ in Ref 2 [Compare Eq (23) with Eqs (15b) and (18a)] For larger ϵ and ϵx, the comparison is more complex and requires numerical computation In particular, for large ϵ, one should integrate Eq (8) numerically rather than use the analytic perturbation solution developed herein Also, if the minimum-length solution is to be compared with that in Ref 2 for the same inlet and exit conditions (in terms of p_1, u, ρ, $\omega\tau_e$), it is necessary to determine the value of n in Ref 2 such that $B_2 = 1$ Several examples have indicated that the accelerator lengths agree within 10% when n is determined as described in the foregoing However, the distribution of jE_yA, A, and B was different for the two solutions A more detailed comparison is beyond the scope of the present note

The minimum-length accelerator solution developed herein, and in Ref 1, is based upon the assumptions of a perfect gas and constant enthalpy, B and σ If these assumptions were relaxed, particularly those regarding the constant enthalpy and B, different minimum lengths would result Physical considerations, such as Hall effects, ion slip, and the need for relatively uniform area variations, may provide other guide lines for minimizing Eq (7) Note that the area variation in the present solution [Eq (17a)] may not permit shock-free supersonic flow Hence, the present solution and that of Ref 1 provide only a first estimate for the design of a physically realistic "minimum-length" accelerator with relatively uniform enthalpy and B It may be necessary to modify this design and then to determine the performance by integrating the full one-dimensional equations of motion

References

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Instability of Three-Row Vortex Streets

CHEE TUNG*
Polytechnic Institute of Brooklyn, Farmingdale, N Y

CLOUD patterns resembling vortices on the leeward side of islands have been observed [1] It is well known that two rows of vortex streets, known as Kármán vortex streets, exist behind a single bluff body representing a highly elevated island in a uniform stream On the leeward side of a group island, it might be expected that more than two rows of vortex streets may exist It is of interest, therefore, to study the possible stable configuration of three rows of vortex streets and their stability criteria The three rows of vortex streets may conceivably appear on the leeward side of two islands with small openings between them It will be shown in this note that there exists a steady configuration of three-row vortex streets, and it is always unstable because a special small disturbance can be found which makes the configuration unstable

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* Research Fellow, Graduate Center

Steady-State Solution

Consider three parallel infinite rows of the same spacing, say a, with a total strength equal to zero, so arranged that the origin of the coordinates is chosen at any midpoint of two vortices in the second row, and that the x axis coincides with that row The coordinates of each vortex in the first, second, and third rows will be $(ma + b, h_1)$, $[na + (a/2), 0]$ and $(pa + c, -h_2)$, respectively, where m, n, and $p = 0, \pm 1, \pm 2,$ The strengths of the three rows of vortices are Γ_1, $-(\Gamma_1 + \Gamma_2)$, Γ_2, respectively The sign of Γ_1 and Γ_2 are arbitrary, but the distances h_1 and h_2 are positive with the middle row chosen as x axis The velocity of vortex[2] at point $z_1 = b + ih_1$, $z_2 = a/2$, and $z_3 = c - ih_2$ may be written as

$$v_1 = -i(\Gamma_1 + \Gamma_2)\frac{\pi}{a}\tan\frac{\pi}{a}(b + h_1 i) - i\Gamma_2\frac{\pi}{a}\cot\frac{\pi}{a}[b - c + (h_1 + h_2)i]$$

$$v_2 = i\Gamma_1\frac{\pi}{a}\tan\frac{\pi}{a}(b + h_1 i) - i\Gamma_2\frac{\pi}{a}\tan\frac{\pi}{a}(c - ih_2)$$

$$v_3 = i\Gamma_1\frac{\pi}{a}\cot\frac{\pi}{a}[c - b - (h_1 + h_2)i] + i(\Gamma_1 + \Gamma_2)\frac{\pi}{a}\tan\frac{\pi}{a}(-c + h_2 i)$$

The steady-state solution requires that $v_1 = v_2 = v_3$, i e ,

$$\tan\frac{\pi}{a}(b + h_1 i) + \cot\frac{\pi}{a}[b - c + (h_1 + h_2)i] = \tan\frac{\pi}{a}(c - h_2 i) \quad (1)$$

$$\tan\frac{\pi}{a}(b + h_1 i) = \cot\frac{\pi}{a}[c - b - (h_1 + h_2)i] - \tan\frac{\pi}{a}(-c + h_2 i) \quad (2)$$

Note that Eqs (1) and (2) are the same To find the values of b and c, we separate Eq (1) into real and imaginary parts [3] Thus they are the following cases

Case 1: b = 0, c = 0

Equation (1) reduces to

$$\tanh\frac{h_1\pi}{a} - \coth\frac{(h_1 + h_2)\pi}{a} = -\tanh\frac{h_2\pi}{a} \quad (3)$$

Let $\tanh(h_1\pi/a) = A$ and $\tanh(h_2\pi/a) = B$; then $0 < A < 1$, $0 < B < 1$ for $h_1 > 0$, $h_2 > 0$ Equation (3) gives

$$A^2 + B^2 = 1 - AB \quad (4a)$$

or

$$B = [-A \pm (4 - 3A^2)^{1/2}]/2 \quad (4b)$$

Since $0 < A < 1$, $A < 1 < (4 - 3A^2)^{1/2}$, then the only root of Eq (4b) such that $0 < B < 1$ will be $[-A \pm (4 - 3A^2)^{1/2}]/2$

Case 2: b = a/2, c = 0

Equation (1) reduces to

$$-(1/A) + [(A + B)/(AB + 1)] = B \quad (5)$$

Therefore, $B = -1 \pm (4A^2 - 3)^{1/2}/2A$ The roots of B in Eq (5) are both negative because $(4A^2 - 3)^{1/2} < 1$ Hence, no steady solution exists for $b = a/2$, $c = 0$, since we pick h_2 as positive

Case 3: b = 0, c = a/2

Equation (1) reduces to $-A + [(A + B)/(1 + AB)] = 1/B$ This is the same as Eq (5) if we interchange A and B; hence, a steady-state solution does not exist